JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 29, No. 6, November–December 2006

Engineering Notes

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Parametric Optimization Analysis for Minimum-Fuel Low-Thrust Coplanar Orbit Transfer

Yumin Chen* and Donglong Sheu†
National Cheng-Kung University, Tainan 70101, Taiwan, Republic of China

DOI: 10.2514/1.21113

Introduction

It is not unusual that a spacecraft is required to make some orbit transfers that are very fuel consuming. Moreover, after launch, the fuel of the spacecraft cannot be replenished in space. How to minimize the fuel consumed during an orbit transfer is therefore of paramount importance.

For several decades, advanced propulsion technologies have been investigated and developed for use in space. These new systems can achieve very high specific impulses while having relatively low thrust. Low-thrust propulsion systems offer several advantages over the more conventional high-thrust systems, including increased performance that may lead to smaller spacecraft [1]. Accordingly, much effort [2–4] has been put into the investigations of low-thrust orbit transfers in recent years.

Optimization analysis for low-thrust trajectory problems, however, is difficult in part because the total transfer time is long [5]. Another problem is that the low-thrust propulsion system is inevitably of the continuous-thrust type. The analysis for the continuous-thrust problems is more difficult because the trajectory cannot be propagated forward without numerical integration [6].

Basically, there are two kinds of numerical methods for the analysis of optimal trajectories, namely, indirect and direct methods [6]. Indirect methods use the calculus of variations to derive the necessary conditions of optimality, that is, the Euler–Lagrange equations, and the transversality conditions. To solve these kind of problems, the gradient method (or the steepest descent method) which is an improved version of the shooting method is commonly used. Direct methods [4], on the other hand, discretize the original problem and transform it into a parameter optimization problem [6]. By using this method, the first-order necessary conditions derived by the calculus of variations are ignored and explicit integration of the system differential equations is avoided. The resulting nonlinear programming problem often consists of a large number of variables and constraints [7]. Although direct methods are robust in the analysis, they may not give a guarantee of optimality because they do not use the adjoint variables to determine the control law [8].

In this study, the optimal trajectories for coplanar circular-to-circular orbit transfer with a continuous constant thrust of which the direction is controllable are solved. To circumvent the complexity of the purely numerical methods, an effective parametric optimization method is proposed. The effectiveness of using this method lies in the accuracy of defining the model of control. To determine the model of control, the second-order gradient method is used to determine the solution first. The second-order gradient method is theoretically elegant but has certain limitations in many cases. By using this method, the extreme solutions are often very sensitive to small changes in the unspecified boundary conditions as a direct result of the nature of the Euler–Lagrange equations (e.g., see [9], p. 214). This is particularly true if the period of integration is long. In that case, the chances of getting divergence increase tremendously.

After a careful study using the second-order gradient method on the optimal orbit transfer problems, it is found that the time histories of optimal control have some patterns. With these patterns, the model of control is assumed with some parameters being left to be determined. The parameters to be determined include the time at which the control is switched. How to accurately determine the influence of the variation of the switching time on the final conditions poses great challenges in this study. To understand the accuracy of the developed parametric optimization method, the results of the analysis are compared with those obtained by using the second-order gradient method.

Equations of Motion

Consider a spacecraft that is acted upon by a thrust in the Earth's gravitational field as shown in Fig. 1. The Earth is assumed to be a perfect sphere and its density dependent only on its radius. The thrust is assumed to be constant and small and its direction controllable. Let the angle between the thrust T and the local horizon be φ as shown in Fig. 1. Then the equations of motion for the spacecraft can be derived as follows (see, e.g., [9], pp. 66–69):

$$\dot{r} \triangleq f_r = u \tag{1a}$$

$$\dot{u} \triangleq f_u = \frac{v^2}{r} - \frac{\mu}{r^2} + \frac{T\sin\varphi}{m} \tag{1b}$$

$$\dot{v} \triangleq f_v = -\frac{uv}{r} + \frac{T\cos\varphi}{m} \tag{1c}$$

$$\dot{m} \triangleq f_m = -\frac{T}{g_0 i_{sp}} \tag{1d}$$

where r is the radial distance from the center of the Earth to the spacecraft, u the radial speed, v the circumferential speed, and m the mass. In these four equations, T is assumed to be constant and the angle, φ $(0 \le \varphi < 2\pi)$, is the control variable. Also, μ is the gravitational parameter of the Earth, i_{sp} the specific impulse, and g_0 the gravitational acceleration on the Earth's surface.

Received 14 November 2005; revision received 14 June 2006; accepted for publication 16 June 2006.

*Research Assistant, Institute of Aeronautics and Astronautics.
†Associate Professor, Institute of Aeronautics and Astronautics, donglong@mail.ncku.edu.tw. Member AIAA.

In this study, the target orbit is assumed to be circular and therefore the terminal conditions can be simply specified by $r(t_f) = r_f$, $u(t_f) = 0$, and $v(t_f) = \sqrt{\mu/r_f}$, where r_f is the radius of the target orbit and t_f is the final time.

Solutions by Using the Second-Order Gradient Method

To establish a rigorous control model, in this study, the optimal trajectories for the transfer between two orbits are solved first by using the second-order gradient method (SOGM). Although the iteration algorithm for the second-order gradient method is quite standard (e.g., [9], pp. 228–234), the initial guesses for a set of qualified control variable histories, final time, and penalty coefficients of constraints to start the iterations which will eventually lead to convergence are in general very difficult, since they must make the corresponding trajectory simultaneously satisfy the convexity condition, the normality condition, and the no-conjugate-point condition.

Usually, the longer the time of flight or the tighter the terminal conditions, the more difficult the initial guesses are to make the corresponding trajectory simultaneously satisfy the three conditions. Based on the concept of the so-called "terminal increment method" [10], it is found that the terminal conditions of the original problem can be modified to make the time of flight shorter so as to make the initial guesses easier. Another good approach is to release some of the terminal constraints so that they can be more easily satisfied.

For different problems, the method of terminal increment varies. To make the problem even easier, the terminal constraint, $u(t_f) = 0$, may be released. Once an optimal trajectory to satisfy the released terminal constraints is successfully determined, the control variable histories, the terminal time, and the penalty coefficients of constraints can be used as a set of initial guesses for a new problem in which the terminal constraint, $u(t_f) = 0$, is reinstated. Again, once an optimal trajectory to transfer between the two close orbits is successfully determined, the corresponding control variable histories, the terminal time, and the penalty coefficients of constraints can be used as a set of initial guesses for a new problem in which the terminal radius is increased.

Here a concept must be clarified. When the method of terminal-radius increment is used, it does not mean that the final solution is obtained by piecewise connecting those obtained in each incremental step. In fact, the solution obtained in each step is used only as a reference trajectory for the solution in the next step.

Parametric Optimization Method with a Piecewise Continuous Control Model

Consider a dynamic system which is governed by a set of differential equations,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{2}$$

where $\mathbf{x} \in R^n$ represent the n state variables, and $\mathbf{u} \in R^m$ the m control variables. A standard optimal control problem is how to determine a set of control functions $\mathbf{u}(t)$ $(0 \le t \le t_f)$, to make a cost, $J = J[\mathbf{x}(t_f), t_f]$, minimum or maximum subject to the terminal constraints

$$\mathbf{g}[\mathbf{x}(t_f), t_f] = \mathbf{0} \tag{3}$$

where $\mathbf{g} \in R^{n_c}$ with n_c being the number of constraints.

From the results obtained with the second-order gradient method, it is found that the control histories for all the cases studied appear to have some trends. Based on the observation, the control functions can be proposed as follows:

$$\mathbf{u}(t) = \begin{cases} \mathbf{u}_1(\mathbf{a}, t), & \text{if } t_0 \le t < t_1 \\ \mathbf{u}_2(\mathbf{b}, t), & \text{if } t_1 < t \le t_f \end{cases} \tag{4}$$

where $\mathbf{u}_1(\mathbf{a}, t)$ and $\mathbf{u}_2(\mathbf{b}, t)$ are continuous functions with $\mathbf{a} \in R^{n_a}$ and $\mathbf{b} \in R^{n_b}$ being vectors with elements of constant parameters to be determined. Denote

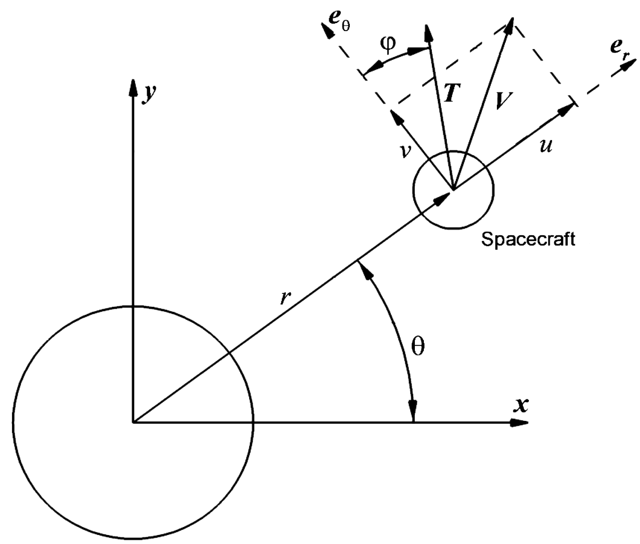

Fig. 1 Definitions of the inertial and moving coordinate systems.

$$\mathbf{p} = \begin{Bmatrix} \mathbf{a} \\ \mathbf{b} \end{Bmatrix} \in R^{n_a + n_b}, \qquad \boldsymbol{\eta} = \begin{Bmatrix} t_1 \\ t_2 \end{Bmatrix} \in R^2 \tag{5}$$

Then the problem becomes to determine $\mathbf{p}$ plus $\boldsymbol{\eta}$, to optimize the cost J, subject to the constraints, Eq. (3). At this point, it is convenient to define the total number of elements in $\mathbf{p}$ as n_p, that is, $n_p \triangleq n_a + n_b$. Because the terminal constraints must be satisfied, the total number of free parameters reduces to $n_p + 2 - n_c$ (which, of course, must be greater than 0). It is well known that, to "free" the constraints, the cost may be augmented to

$$\bar{J} \triangleq J + \boldsymbol{\nu}^T \mathbf{g} \tag{6}$$

where $\boldsymbol{\nu} \in R^{n_c}$ is known as the vector of penalty coefficients of constraints. With $\bar{J}$ being defined in Eq. (6), all $\mathbf{p}$ plus $\boldsymbol{\eta}$ can be considered as free parameters. To optimize $\bar{J}$ with respect to $\mathbf{p}$ and $\boldsymbol{\eta}$, the necessary conditions are

$$\bar{J}_{\mathbf{p}}^T = \mathbf{x}_{\mathbf{p}}^T(t_f) \bar{J}_{\mathbf{x}(t_f)}^T = \mathbf{0} \tag{7a}$$

$$\bar{J}_{t_1} = \bar{J}_{\mathbf{x}(t_f)} \mathbf{x}_{t_1}(t_f) + \bar{J}_{\bar{t}_1} = 0 \tag{7b}$$

$$\bar{J}_{t_f} = \bar{J}_{\mathbf{x}(t_f)} \mathbf{x}_{t_f}(t_f) + \bar{J}_{\bar{t}_f} = 0 \tag{7c}$$

where $\bar{J}_{t_1}$ and $\bar{J}_{\bar{t}_1}$ represent, respectively, the implicit and explicit partial differentiations of $\bar{J}$ with respect to t_1. The same convention applies to $\bar{J}_{t_f}$ and $\bar{J}_{\bar{t}_f}$.

In Eq. (7a), $\mathbf{x}_{\mathbf{p}}(t_f)$ may be determined by integrating from 0 to t_f the following equation:

$$\dot{\mathbf{x}}_{\mathbf{p}} = \mathbf{f}_{\mathbf{x}} \mathbf{x}_{\mathbf{p}} + \mathbf{f}_{\mathbf{u}} \mathbf{u}_{\mathbf{p}}, \qquad (0 \le t \le t_f), \qquad \mathbf{x}_{\mathbf{p}}(0) = \mathbf{0} \tag{8a}$$

which is a variation of Eq. (2) with respect to the parameters $\mathbf{p}$. In Eq. (7b), $\mathbf{x}_{t_1}(t_f)$ can be determined by integrating

$$\dot{\mathbf{x}}_{t_1} = \mathbf{f}_{\mathbf{x}} \mathbf{x}_{t_1} + \mathbf{f}_{\mathbf{u}_2} \mathbf{u}_{2t_1}, \qquad (t_1 \le t \le t_f) \tag{8b}$$

from t_1 to t_f with the condition at t_1,

$$\mathbf{x}_{t_1}(t_1) = \mathbf{f}[\mathbf{x}(t_1), \mathbf{u}_1(t_1)] - \mathbf{f}[\mathbf{x}(t_1), \mathbf{u}_2(t_1)]$$

while in Eq. (7c), $\mathbf{x}_{t_f}(t_f)$ can be simply determined from

$$\mathbf{x}_{t_f}(t_f) = \mathbf{f}[\mathbf{x}(t_f), \mathbf{u}_2(t_f)]$$

Now the problem becomes to solve Eqs. (7a–7c) for $\mathbf{p}$, $\boldsymbol{\eta}$, and $\boldsymbol{\nu}$. There are totally $n_p + 2 + n_c$ equations which can be used to solve for the same number of unknown parameters. These are a set of highly nonlinear simultaneous algebraic equations. To solve this problem, the parameters $\mathbf{p}$, $\boldsymbol{\eta}$, and $\boldsymbol{\nu}$ must be guessed initially. Of course, for a set of arbitrarily guessed parameters, Eqs. (7a–7c) may not be satisfied. To improve the estimation of the parameters, it is required to determine how the necessary conditions vary with the parameters. Consider that if the parameters have the variations, $\Delta\mathbf{p}$, $\Delta\boldsymbol{\eta}$, and $\Delta\boldsymbol{\nu}$, then the functions, $\bar{J}_{\mathbf{p}}$, $\bar{J}_{\boldsymbol{\eta}}$, and $\mathbf{g} = \bar{J}_{\boldsymbol{\nu}}^T$, will have the variations

$$\Delta\bar{J}_{\mathbf{p}}^T = \bar{J}_{\mathbf{pp}}\Delta\mathbf{p} + \bar{J}_{\mathbf{p}\boldsymbol{\eta}}\Delta\boldsymbol{\eta} + \bar{J}_{\mathbf{p}\boldsymbol{\nu}}\Delta\boldsymbol{\nu} \tag{9a}$$

$$\Delta\bar{J}_{\boldsymbol{\eta}}^T = \bar{J}_{\boldsymbol{\eta}\mathbf{p}}\Delta\mathbf{p} + \bar{J}_{\boldsymbol{\eta}\boldsymbol{\eta}}\Delta\boldsymbol{\eta} + \bar{J}_{\boldsymbol{\eta}\boldsymbol{\nu}}\Delta\boldsymbol{\nu} \tag{9b}$$

$$\Delta\mathbf{g} = \Delta\bar{J}_{\boldsymbol{\nu}}^T = \bar{J}_{\boldsymbol{\nu}\mathbf{p}}\Delta\mathbf{p} + \bar{J}_{\boldsymbol{\nu}\boldsymbol{\eta}}\Delta\boldsymbol{\eta} \tag{9c}$$

correspondingly, where

$$\bar{J}_{\mathbf{pp}} \triangleq (\bar{J}_{\mathbf{p}}^T)_{\mathbf{p}}, \qquad \bar{J}_{\mathbf{p}\boldsymbol{\eta}} \triangleq (\bar{J}_{\mathbf{p}}^T)_{\boldsymbol{\eta}}, \qquad \bar{J}_{\boldsymbol{\nu}\mathbf{p}} \triangleq (\bar{J}_{\boldsymbol{\nu}}^T)_{\mathbf{p}} = \mathbf{g}_{\mathbf{p}}, \ldots$$

By choosing

$$\Delta\bar{J}_{\mathbf{p}}^T = -\varepsilon\bar{J}_{\mathbf{p}}^T, \qquad \Delta\bar{J}_{\boldsymbol{\eta}}^T = -\varepsilon\bar{J}_{\boldsymbol{\eta}}^T, \qquad \Delta\mathbf{g} = \Delta\bar{J}_{\boldsymbol{\nu}}^T = -\varepsilon\mathbf{g}$$

where $0 < \varepsilon \leq 1$, $\Delta\mathbf{p}$, $\Delta\boldsymbol{\eta}$, and $\Delta\boldsymbol{\nu}$ can be determined by solving Eqs. (9a–9c) simultaneously.

Now how to determine the second-order derivatives which exist implicitly in Eqs. (9a–9c)? Consider that Eqs. (8a) and (8b) in fact can also be written as follows:

$$\dot{\mathbf{x}}_{p_i} = \sum_{k=1}^{n}\mathbf{f}_{x_k}x_{k,p_i} + \sum_{k=1}^{m}\mathbf{f}_{u_k}u_{k,p_i}, \qquad (i = 1, 2, \ldots, n)$$

$$\dot{\mathbf{x}}_{t_1} = \sum_{k=1}^{n}\mathbf{f}_{x_k}x_{k,t_1} + \sum_{k=1}^{m}\mathbf{f}_{u_k}u_{k,t_1}$$

Differentiating the above two equations with respect to p_j ($j = 1, 2, \ldots, n$) and t_1 gives

$$\dot{\mathbf{x}}_{p_ip_j} = \sum_{k=1}^{n}\left(\sum_{l=1}^{n}\mathbf{f}_{x_kx_l}x_{l,p_j} + \sum_{l=1}^{m}\mathbf{f}_{x_ku_l}u_{l,p_j}\right)x_{k,p_i} + \sum_{k=1}^{n}\mathbf{f}_{x_k}x_{k,p_ip_j} + \sum_{k=1}^{m}\left(\sum_{l=1}^{n}\mathbf{f}_{u_kx_l}x_{l,p_j} + \sum_{l=1}^{m}\mathbf{f}_{u_ku_l}u_{l,p_j}\right)u_{k,p_i} + \sum_{k=1}^{m}\mathbf{f}_{u_k}u_{k,p_ip_j} \tag{10a}$$

$$(\text{for } 0 \leq t \leq t_f \quad \text{and} \quad i, j = 1, 2, \ldots, n)$$

$$\dot{\mathbf{x}}_{p_it_1} = \sum_{k=1}^{n}\left(\sum_{l=1}^{n}\mathbf{f}_{x_kx_l}x_{l,t_1} + \sum_{l=1}^{m}\mathbf{f}_{x_ku_l}u_{l,t_1}\right)x_{k,p_i} + \sum_{k=1}^{n}\mathbf{f}_{x_k}x_{k,p_it_1} + \sum_{k=1}^{m}\left(\sum_{l=1}^{n}\mathbf{f}_{u_kx_l}x_{l,t_1} + \sum_{l=1}^{m}\mathbf{f}_{u_ku_l}u_{l,t_1}\right)u_{k,p_i} + \sum_{k=1}^{m}\mathbf{f}_{u_k}u_{k,p_it_1} \tag{10b}$$

$$(\text{for } t_1 < t \leq t_f \quad \text{and} \quad i = 1, 2, \ldots, n)$$

$$\dot{\mathbf{x}}_{t_1t_1} = \sum_{k=1}^{n}\left(\sum_{l=1}^{n}\mathbf{f}_{x_kx_l}x_{l,t_1} + \sum_{l=1}^{m}\mathbf{f}_{x_ku_l}u_{l,t_1}\right)x_{k,t_1} + \sum_{k=1}^{n}\mathbf{f}_{x_k}x_{k,t_1t_1} + \sum_{k=1}^{m}\left(\sum_{l=1}^{n}\mathbf{f}_{u_kx_l}x_{l,t_1} + \sum_{l=1}^{m}\mathbf{f}_{u_ku_l}u_{l,t_1}\right)u_{k,t_1} + \sum_{k=1}^{m}\mathbf{f}_{u_k}u_{k,t_1t_1} \tag{10c}$$

$$(\text{for } t_1 < t \leq t_f)$$

Equation (10a) can be integrated from 0 to t_f with the initial conditions $\mathbf{x}_{p_ip_j}(0) = \mathbf{0}$ to determine $\mathbf{x}_{p_ip_j}(t_f)$. At this point, it is worthwhile to mention that for $t < t_1$, $\mathbf{x}_{p_it_1}(t) = \mathbf{0}$ and $\mathbf{x}_{t_1t_1}(t) = \mathbf{0}$. To determine $\mathbf{x}_{p_it_1}(t_f)$ and $\mathbf{x}_{t_1t_1}(t_f)$, Eqs. (10b) and (10c) can be integrated from t_1 to t_f with the conditions

$$\mathbf{x}_{p_it_1}(t_1) = [\mathbf{f}_{\mathbf{x}}(\mathbf{x}, \mathbf{u}_1)\mathbf{x}_{p_i} + \mathbf{f}_{\mathbf{u}_1}(\mathbf{x}, \mathbf{u}_1)\mathbf{u}_{1,p_i}]_{t_1^-} - [\mathbf{f}_{\mathbf{x}}(\mathbf{x}, \mathbf{u}_2)\mathbf{x}_{p_i} + \mathbf{f}_{\mathbf{u}_2}(\mathbf{x}, \mathbf{u}_2)\mathbf{u}_{2,p_i}]_{t_1^+}$$

($i = 1, 2, \ldots, n$) and

$$\mathbf{x}_{t_1t_1}(t_1) = [\mathbf{f}_{\mathbf{x}}(\mathbf{x}, \mathbf{u}_1)\mathbf{f}(\mathbf{x}, \mathbf{u}_1) + \mathbf{f}_{\mathbf{u}_1}(\mathbf{x}, \mathbf{u}_1)\mathbf{u}_{1,t_1}]_{t_1^-} - [\mathbf{f}_{\mathbf{x}}(\mathbf{x}, \mathbf{u}_2)\mathbf{f}(\mathbf{x}, \mathbf{u}_1) + \mathbf{f}_{\mathbf{u}_2}(\mathbf{x}, \mathbf{u}_2)\mathbf{u}_{2,t_1}]_{t_1^+} - [\mathbf{f}_{\mathbf{x}}(\mathbf{x}, \mathbf{u}_2)\mathbf{x}_{t_1}(t_1) + \mathbf{f}_{\mathbf{u}_2}(\mathbf{x}, \mathbf{u}_2)\mathbf{u}_{2,t_1}]_{t_1^+}$$

at t_1, respectively. Also, by substituting $t = t_f$ in Eqs. (8a) and (8b), respectively, $\mathbf{x}_{p_it_f}$ and $\mathbf{x}_{t_1t_f}$ can be obtained as follows:

$$\mathbf{x}_{p_it_f}(t_f) = [\mathbf{f}_{\mathbf{x}}\mathbf{x}_{p_i} + \mathbf{f}_{\mathbf{u}_2}\mathbf{u}_{2,p_i}]_{t_f} \tag{10d}$$

$$\mathbf{x}_{t_1t_f}(t_f) = [\mathbf{f}_{\mathbf{x}}\mathbf{x}_{t_1} + \mathbf{f}_{\mathbf{u}_2}\mathbf{u}_{2,t_1}]_{t_f} \tag{10e}$$

Finally, $\mathbf{x}_{t_ft_f}$ can be determined by differentiating Eq. (2) with respect to t and then substituting t_f in the resulted equation. Accordingly,

$$\mathbf{x}_{t_ft_f}(t_f) = [\mathbf{f}_{\mathbf{x}}\mathbf{x}_{t_f} + \mathbf{f}_{\mathbf{u}_2}\mathbf{u}_{2,t_f}]_{t_f} \tag{10f}$$

In summary, the algorithm to compute the parameters $\mathbf{p}$, $\boldsymbol{\eta}$, and $\boldsymbol{\nu}$ and the corresponding trajectory is as follows:

1) Guess a set of $\mathbf{p}$, $\boldsymbol{\eta}$, and $\boldsymbol{\nu}$.
2) Simultaneously integrate Eqs. (2), (8a), and (10a) from 0 to t_f and integrate Eqs. (8b), (10b), and (10c) from t_1 to t_f.
3) Check if $\bar{J}_{\mathbf{p}}$, $\bar{J}_{\boldsymbol{\eta}}$, and $\mathbf{g}$ are all less than the preset values. If yes, then stop; otherwise continue on with the next step.
4) Solve Eqs. (9a–9c) simultaneously for $\Delta\mathbf{p}$, $\Delta\boldsymbol{\eta}$, and $\Delta\nu$.
5) Modify $\mathbf{p}$, $\boldsymbol{\eta}$, and $\boldsymbol{\nu}$ with $\mathbf{p} + \Delta\mathbf{p}$, $\boldsymbol{\eta} + \Delta\boldsymbol{\eta}$, and $\boldsymbol{\nu} + \Delta\boldsymbol{\nu}$, respectively.
6) Go to step 2.

Solutions by Using the Parametric Optimization Methods

To illustrate the theory more explicitly, consider that a spacecraft with initial mass $m_0 = 500$ kg is propelled with constant thrust. The problem is to determine the minimum-fuel transfer between two coplanar circular orbits. By referring to the coordinate system defined in Fig. 1, the initial conditions, $r_0 = 2$ DU, $u_0 = 0$ DU/TU, and $v_0 = \sqrt{\mu/r_0}$ and the final conditions, $r_f = 2.2$ DU, $u_f = 0$ DU/TU, and $v_f = \sqrt{\mu/r_f}$ are given, where DU and TU are the canonical distance and time unit [11], respectively. Here 1 DU is defined as the length of the radius of the Earth's equator and 1 DU/TU is defined as the circular orbit speed with the radius of 1 DU. From both definitions, the time units 1 TU can be derived. To solve this problem, two parametric methods with simple control functions are introduced, namely, the piecewise constant control method (PCCM) and the piecewise linear-time-function control method (PLCM).

For PCCM, the control of the thrust direction is defined by the piecewise constant function as follows:

$$\mathbf{u} = \varphi = \begin{cases} a_0, & \text{if } 0 \leq t < t_1 \\ b_0, & \text{if } t_1 < t \leq t_f \end{cases} \tag{11}$$

where a_0 and b_0 are constants to be determined. With this definition, the parameters described in the format of Eq. (5) and the time parameters become

$$\mathbf{p} = \begin{bmatrix} a_0 & b_0 \end{bmatrix}^T \quad \text{and} \quad \boldsymbol{\eta} = \begin{bmatrix} t_1 & t_f \end{bmatrix}^T$$

respectively. The number of the parameters is $n_p = 2$.

To improve the accuracy of the solution, for PLCM, the piecewise constant function is modified to a piecewise linear-time function as follows:

$$\mathbf{u} = \varphi = \begin{cases} a_0 + a_1 t, & \text{if } 0 \le t < t_1 \\ b_0 + b_1 t, & \text{if } t_1 < t \le t_f \end{cases} \tag{12}$$

where a_0, a_1, b_0, and b_1 are constants to be determined. With this definition, the parameters described in the format of Eq. (5) become

$$\mathbf{p} = \begin{bmatrix} a_0 & a_1 & b_0 & b_1 \end{bmatrix}^T \quad \text{and} \quad \boldsymbol{\eta} = \begin{bmatrix} t_1 & t_f \end{bmatrix}^T$$

The number of the parameters for this case is $n_p = 4$.

For smaller thrust, the transfer time naturally becomes longer. Also, if the specific impulse i_{sp} of the rocket is too small, the fuel consumption rate becomes too large. After extensive studies using the gradient method, it is found that if the thrust-to-weight ratio $T/W_0 < 5/100$, then the integration time becomes so large that divergence of computation arises inevitably. In this study, it is assumed that $T/W_0 = 5/100$ and $i_{\text{sp}} = 600$ s.

By using the SOGM, the PCCM, and the PLCM, it is possible to obtain the near optimal transfer trajectories from the circular orbit with $r_0 = 2$ DU to the coplanar circular target orbits with $r_f = 4$ DU and 6 DU as shown in Figs. 2–5. In these figures, the trajectories computed by using the SOGM are represented with the solid lines, while the trajectories computed by using the PCCM and PLCM are represented with the dashed and the dash-dotted lines, respectively. For the cases in which the radii of the target orbits are $r_f = 4$ DU and 6 DU, the curves are distinguished with the square and the diamond symbols, respectively.

Figure 2 shows the relationship between the radial distance r and time t, for all cases computed by using the SOGM, PCCM, and PLCM. From the radial distances plotted in this figure, it is found that, with the results obtained by using the SOGM as standards, the results obtained by using the PLCM can better characterize the optimal trajectory than those obtained by using the PCCM.

Figure 3 shows the relationship between the radial speed u and time t. In this figure, it is found that the radial speed computed by using the PCCM is a little smaller in the first phase but is a little larger in the third phase than those computed by using the SOGM. The switching time from the first phase to the third phase computed by using the PCCM is a little later than those computed by using the SOGM. Also, the maximum radial speed computed by using the PCCM is a little smaller that those computed by using the SOGM. From the figure, it is found that, however, the errors of the results obtained by using the PLCM are almost negligible as compared with those obtained by using the SOGM.

Figure 4 shows the relationship between the circumferential speed v and time t. From this figure, it is very clear that the circumferential speeds computed by using the PCCM have very large errors as compared with those computed by using the SOGM. The results computed by using the PCCM are underestimated in the first 40% of transfer time and are overestimated after that time. The circumferential speed computed by using the PCCM is minimum at the final time instead of being somewhere before the final time. Obviously, the results computed by using the PCCM cannot accurately display the characteristics of circumferential speed as computed by using the SOGM. These drawbacks can be tremendously improved by using the PLCM.

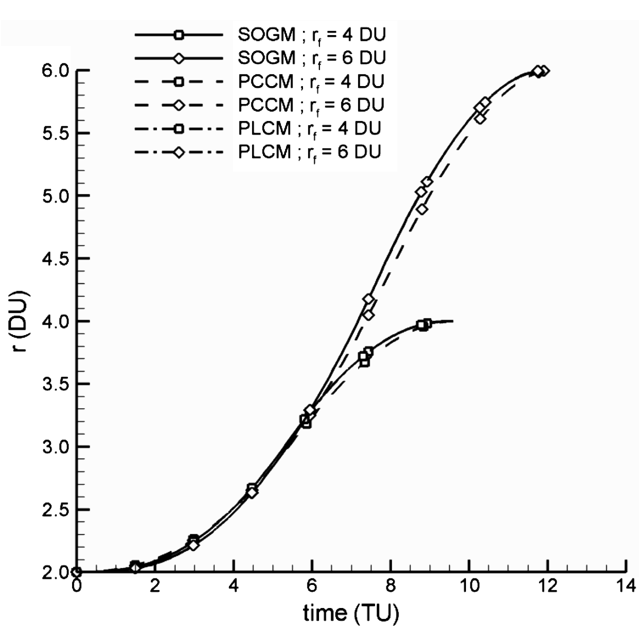

Fig. 2 Comparison of the radial distances computed by using the PCCM, the PLCM, and the SOGM.

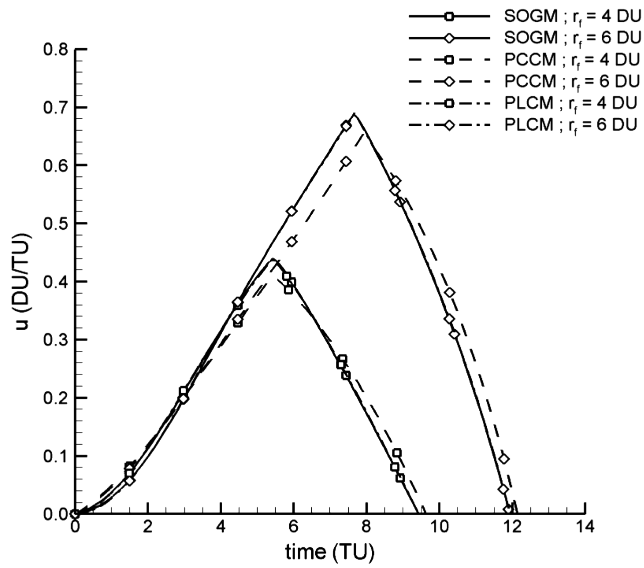

Fig. 3 Comparison of the radial speeds computed by using the PCCM, the PLCM, and the SOGM.

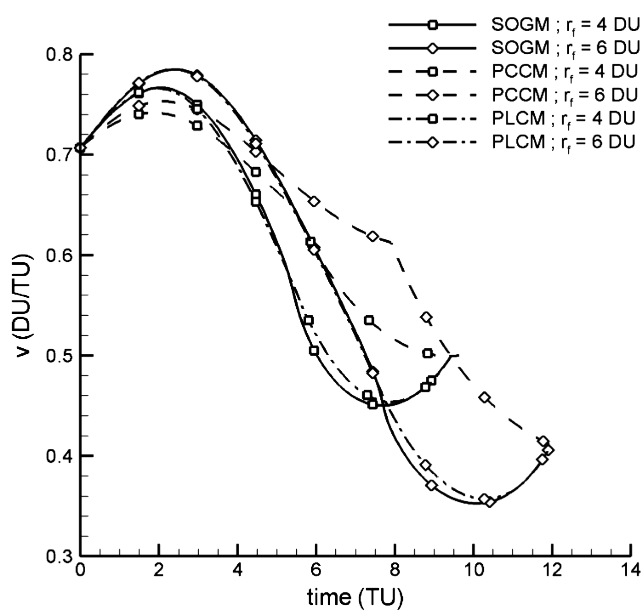

Fig. 4 Comparison of the circumferential speeds computed by using the PCCM, the PLCM, and the SOGM.

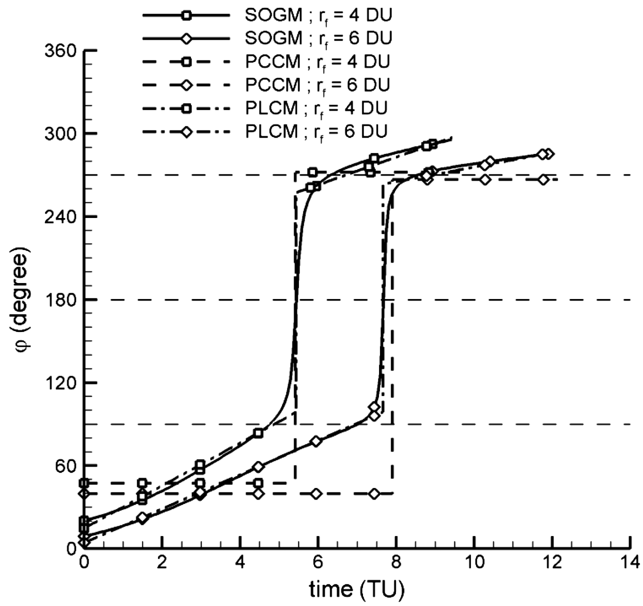

Fig. 5 Comparison of the control variables computed by using the PCCM, the PLCM, and the SOGM.

Figure 5 shows the relationship between the control angle φ and time t. In this figure, it is observed that, for the case of which the target radius is $r_f = 4$ DU, the switching times t_1 computed by using all the PCCM, the PLCM, and the SOGM are nearly at the midpoint of the transfer time. The approximation is indeed very good for this case. However, for the case where the target radius is $r_f = 6$ DU, the switching time t_1 computed by the PCCM is a little late as compared with that computed by using the SOGM. In contrast to those obtained by using the PCCM, the results obtained by using the PLCM can obviously better characterize the optimal control angle as obtained by using the SOGM. This is true not only for the switching time t_1. As can be found from the figure, for both cases, the control functions computed by using the PLCM are indeed very close to those computed by using the SOGM, whereas the control functions computed by using the PCCM show large deviations.

Conclusions

In this study, a model of piecewise continuous time functions is successfully applied to approximate the optimal thrust control angle for minimum-fuel coplanar orbit transfers with constant low thrust from a near-Earth circular orbit to some circular target orbits. After extensive studies, it is found that, by assuming the control angle as two continuous time functions plus a switch at time t_1, the optimal transfer trajectories can be fairly well characterized. Before t_1, the thrust control angle is smaller than 90 deg. It means that the thrust makes the spacecraft accelerate in both the radial and the circumferential directions in this period. At t_1, the thrust control angle quickly switches to an angle in between 270 deg and 360 deg. It means that the thrust makes the spacecraft decelerate in the radial direction while accelerating in the circumferential direction. As compared with the fuel consumptions obtained by using the SOGM, those obtained by using the PCCM are about 2% more whereas those obtained by using the PLCM are only 0.09% more. The principal result here is that only a few parameters are required to approximate the optimal transfer trajectory well. This largely reduces the complexity of the control function as compared to that generated by the second-order gradient method.

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